

SMASH NILPOTENT CYCLES ON PRODUCTS OF CURVES

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ABSTRACT. Voevodsky has conjectured that numerical and smash equivalence coincide on a smooth projective variety. We prove the conjecture for one dimensional cycles on an arbitrary product of curves. As a consequence we get that numerically trivial 1-cycles on an abelian variety are smash nilpotent.

1. INTRODUCTION

Throughout this article we work over an algebraically closed field. Let X be a smooth and projective variety. Define rational cycles of codimension n to be elements of the group $Z^n(X) := \bigoplus_{Y \subset X} \mathbb{Q} \cdot [Y]$, where Y varies over the set of irreducible closed subvarieties of X of codimension n . Various equivalence relations may be defined on the group of cycles. Examples of such equivalence relations are rational equivalence, algebraic equivalence, homological equivalence and numerical equivalence.

Homological equivalence is defined using a Weil cohomology theory. An important consequence of the two standard conjectures (Lefschetz standard conjecture and Hodge standard conjecture) is that numerical and homological equivalence coincide, which would mean that being homologically trivial is independent of the choice of the Weil cohomology theory. For an exposition on the standard conjectures we refer the reader to [Kle94]. In [Kle94], Kleiman writes, “Sometimes, the coincidence of the two relations may be proved by sandwiching homological equivalence between numerical equivalence and another equivalence relation.”

Voevodsky [Voe95] introduced the adequate relation of smash nilpotence. Let X be a smooth projective variety over a field k . An algebraic cycle α on X (with rational coefficients) is *smash-nilpotent* if there exists $n > 0$ such that α^n is rationally equivalent to 0 on X^n . Voevodsky [Voe95, Cor. 3.3] and Voisin [Voi94, Lemma 2.3] proved that any cycle algebraically equivalent to 0 is smash-nilpotent. Because of the multiplicative property of the cycle class map, any smash-nilpotent cycle is homologically equivalent to 0 and so numerically equivalent to 0; Voevodsky conjectured that the converse is true [Voe95, Conj. 4.2]. This would imply that homological and numerical equivalence coincide. This also implies Bloch’s conjecture on zero cycles on surfaces with $p_g = 0$. Moreover, this is an interesting question in itself.

The first general result which gave examples of smash nilpotent cycles is the following result of Kimura, [Kim05, Proposition 6.1]. If M and N be finite dimensional motives with different parity and $f : M \rightarrow N$ a morphism of motives, then f is smash nilpotent.

In [She74], Shermenev proves that, for an abelian variety A , the motive $h^1(A)$ is oddly finite dimensional. In [KS09], the authors combine this with results in [Kim05], to show that skew cycles (β is called *skew* if $[-1]^*\beta = -\beta$) are smash nilpotent. From this, using results in [Bea86], they prove that on an abelian variety of dimension ≤ 3 , any homologically trivial cycle is smash nilpotent. In fact, using the motivic hard Lefschetz theorem for abelian varieties, [Kun93, Theorem 5.2], one can easily improve the above

theorem to equality of numerical and smash equivalence for abelian varieties of dimension ≤ 3 . In particular, Voevodsky's conjecture is true for abelian varieties of dimension ≤ 3 .

Conjecturally, morphisms between motives of the same parity given by a numerically trivial cycle should be smash nilpotent. The author is not aware of any nontrivial examples (that is, not algebraically equivalent to 0 or in the subalgebra generated by skew cycles on an abelian variety under the operations of intersection product and Pontryagin product) or any general results in this direction.

On abelian varieties of dimension ≥ 4 , the Griffiths group of symmetric cycles (β is called *symmetric* if $[-1]^*\beta = \beta$) can have infinite rank, as shown by Fakhruddin in [Fak96, Theorem 4.4]. Symmetric cycles can be viewed as morphisms between motives of the same parity. The methods in [Kim05] and [KS09] do not directly apply to symmetric cycles and one is interested in knowing if these cycles are smash nilpotent. In this article we give the first fairly general results in this direction.

The main theorem in this article is

Theorem. *Voevodsky's conjecture is true for 1-cycles on a product of curves.*

As a consequence of the above, we get the following results

Theorem. *Voevodsky's conjecture is true for 1-cycles on abelian varieties.*

In particular, the cycles constructed by Fakhruddin in [Fak96, Theorem 4.4, Corollary 4.6] are smash nilpotent. It is expected to be true that for any i , we can find g sufficiently large such that the Beauville component $[C]_i$ is non-zero in the Griffiths group of the Jacobian of a generic curve of genus g .

Corollary. *Numerical and smash equivalence coincide on the tautological subring of the Chow ring modulo algebraic equivalence of the Jacobian of a smooth projective curve.*

The proof of theorem 6 proceeds by induction on the number of factors in the product. For any 1-cycle α , we show $\alpha \sim_{sm} \sum \alpha_i$ such that each α_i is obtained in a canonical way from α and comes from a smaller product of curves. To prove this assertion, we are lead to considering 1-cycles on C^m of the type

$$\Delta_e = \sum_{T \neq \emptyset, T \subset S} q_{\#T} \Delta_T$$

Here S denotes the set $\{1, 2, \dots, m\}$ and Δ_T denotes the curve C embedded diagonally into the factor given by T . The above “modified diagonal” cycle is inspired from [GS95]. If m is small, then it is not clear whether this cycle is smash nilpotent. However, for $m \gg 0$, since this is a symmetric cycle and $S^m C$ is a projective bundle over $J(C)$, it is easy to deduce sufficient conditions on the rational coefficients so that

- Δ_e is smash nilpotent.
- Projecting Δ_e to a smaller product C^n yields a nontrivial relation of the type $\alpha \sim_{sm} \alpha_i$.

Solving for the coefficients boils down to showing that certain linear homogeneous polynomials are linearly independent, which is done in section 4.

It was brought to our attention by N. Fakhruddin, that the above method can be used to prove the following interesting proposition. The idea is similar to the above, if we have a symmetric 1-cycle on C^m then we can map it to $J(C)$ in the usual way. Another thing one could do is that we can first project it to C^n for some $n < m$ and

then again push it forward to $J(C)$. Then these two cycles in $J(C)$ can be very different; the first could be something simple (that is, something one understands) whereas the second could be very different, that is, not expressible in some elementary way in terms of the first.

Proposition. *Let C be a smooth projective curve. Denote by $\alpha_j \in CH_j^{g-1}(J(C))/\sim_{alg}$ its Beauville components. If $\alpha_i = 0$ for some $i \geq 0$, then $\alpha_j = 0$ for every $j > i$.*

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2. 1-CYCLES ON PRODUCTS OF CURVES

For a smooth projective curve C , we denote its Jacobian by $J(C)$. In this section we show that numerically trivial 1-cycles on products of smooth projective curves are smash nilpotent.

Let \tilde{D} be a smooth projective curve of genus g and let m be an integer such that $m > 2g + 2$. Let S denote the set $\{1, 2, \dots, m\}$. Let $p_i : \tilde{D}^m \rightarrow \tilde{D}$ denote the projection onto the i -th factor. For every nonempty $T \subset S$ consider the morphism $\phi_T : \tilde{D} \rightarrow \tilde{D}^m$ which is defined by

$$p_i \circ \phi_T(d) = \begin{cases} d & \text{for } i \in T \\ d_0 & \text{for } i \notin T \end{cases}$$

and define Δ_T to be $\phi_{T*}(\tilde{D})$.

The modified diagonal cycle was introduced in [GS95], who show that it is homologically trivial. We define a more general modified diagonal cycle in the Chow group of \tilde{D}^m to be

$$(2.1) \quad \Delta_e = \sum_{T \neq \emptyset, T \subset S} q_{\#T} \Delta_T$$

We want to choose the coefficients $q_i, i = 1, 2, \dots, m$, appropriately so that Δ_e becomes smash nilpotent. Consider the following two conditions

(S1) For every **even** integer i in the set $\{0, 1, 2, \dots, g-1\}$,

$$\left(\sum_{k=1}^m q_k \binom{m}{k} k^{2+i} \right) = 0$$

$$(S2) \quad \sum_{k=1}^m q_k \binom{m}{k} k = 0$$

Remark 1. The above is a homogeneous system of equations in q_k 's and since $m > 2g + 2$, there are more variables than equations. Thus, it will always have nontrivial solutions.

Let $f : \tilde{D}^m \rightarrow S^m \tilde{D}$ denote the quotient by the group S_m , which acts on \tilde{D}^m by permutations. On $S^m \tilde{D}$ consider the (reduced) subvariety $\Delta_i^s := f(\Delta_T)$, for some T with $\#T = i$ (this is clear depends only on $\#T$).

Proposition 2. *We have the following relation in $CH_1(\tilde{D}^m)$*

$$\frac{1}{i!(m-i)!} f^* \Delta_i^s = \sum_{\#T=i} \Delta_T$$

Proof. Since $f : \tilde{D}^m \rightarrow S^m \tilde{D}$ is a finite flat Galois morphism, it is clear that $f^* \Delta_i^s$ is a multiple of $\sum_{\#T=i} \Delta_T$. Applying f_* on both cycles and using the fact that $f_* f^* = m!$ for a finite map, we get the desired result. \square

Since $m > 2g + 2$, $S^m \tilde{D}$ is the projective bundle associated to a locally free sheaf on $J(\tilde{D})$. We may take $\mathcal{O}(1)$ to be the line bundle associated to the (reduced) divisor $f(p_1^{-1}(d_1))$.

Proposition 3. $\deg(f_*(\Delta_T) \cap c_1(\mathcal{O}(1))) = (m-1)!(\#T)$

Proof. It suffices to compute the degree of $\Delta_T \cap f^* c_1(\mathcal{O}(1))$. Arguing as in the proof of proposition 2, we get $f^*(c_1(\mathcal{O}(1))) = (m-1)! \sum_{i=1}^m p_i^{-1}(d_1)$. Taking the set theoretic intersection of Δ_T with $\sum_{i=1}^m p_i^* \mathcal{O}_{\tilde{D}}(d_1)$ gives the desired result. \square

The Beauville decomposition of the Chow group of 1-cycles on $J(\tilde{D})$ is given by (see [Bea86, Proposition 3a])

$$CH_{\mathbb{Q}}^{g-1}(J(\tilde{D})) = \bigoplus_{s=0}^{g-1} CH_s^{g-1}(J(\tilde{D}))$$

$$CH_s^{g-1}(J(\tilde{D})) = \{x \in CH_{\mathbb{Q}}^{g-1}(J(\tilde{D})) \mid [n]_*(x) = n^{2+s}x\}$$

Write the image of the curve embedded into $J(\tilde{D})$ using the base point d_0 as $[\tilde{D}] = \sum_{i=0}^{g-1} \alpha_i$ where α_i satisfies $[n]_* \alpha_i = n^{2+i} \alpha_i$.

Lemma 4. *Let $q_i, i = 1, 2, \dots, m$ be rational numbers satisfying conditions (S1) and (S2). Then the cycle Δ_e in equation (2.1) is smash nilpotent.*

Proof. Define a 1-cycle on $S^m \tilde{D}$ by

$$\Gamma_e := \sum_{k=1}^m q_k \binom{m}{k} \Delta_k^s$$

From proposition 2 it is clear that $f^*(\Gamma_e) = m! \Delta_e$ and it suffices to show that Γ_e is smash nilpotent. Using the base point d_0 get a map $p : S^m \tilde{D} \rightarrow J(\tilde{D})$. Since $m > 2g - 1$, p is a projective bundle associated to a locally free sheaf. In particular, the cycle Γ_e , which is a 1-cycle, may be written as, see [Ful97, Theorem 3.3(b), Proposition 3.1(a)(i)],

$$\Gamma_e = c_1(\mathcal{O}(1))^{m-g-1} \cap p^* \beta_0 \oplus c_1(\mathcal{O}(1))^{m-g} \cap p^* \beta_1$$

In the above equation, $\beta_i \in CH_i(J(\tilde{D}))$ are given by

$$(2.2) \quad \beta_0 = p_*(\Gamma_e \cap c_1(\mathcal{O}(1))) - p_*(c_1(\mathcal{O}(1))^{m-g+1} \cap p^* \beta_1) \quad \text{and} \quad \beta_1 = p_*(\Gamma_e)$$

First we show β_1 is smash nilpotent.

$$\begin{aligned}
\beta_1 &= \sum_{k=1}^m q_k \binom{m}{k} p_*(\Delta_k^s) = \sum_{k=1}^m q_k \binom{m}{k} [k]_*([\tilde{D}]) \\
&= \sum_{k=1}^m q_k \binom{m}{k} [k]_* \left(\sum_{i=0}^{g-1} \alpha_i \right) \\
&= \sum_{i=0}^{g-1} \left(\sum_{k=1}^m q_k \binom{m}{k} k^{2+i} \right) \alpha_i
\end{aligned}$$

Using [KS09, Proposition 1], α_i is smash nilpotent for i odd. Since the q_k 's satisfy (S1), it follows that β_1 is smash nilpotent. In particular, β_1 is numerically trivial.

Next we compute the degree of β_0 . Since β_1 is numerically trivial,

$$\deg(\beta_0) = \deg(p_*(\Gamma_e \cap c_1(\mathcal{O}(1))))$$

A cycle of dimension 0 on an abelian variety is smash nilpotent if and only if its degree is 0. Using proposition 3, it is easily checked that for β_0 to be smash nilpotent, we need that

$$\sum_{k=1}^m q_k \binom{m}{k} k = 0$$

From this, it follows that Γ_e is smash nilpotent and so Δ_e is smash nilpotent. \square

Let $C_i, i = 1, 2, \dots, n$ be smooth, projective curves. Denote by X the product $C_1 \times C_2 \times \dots \times C_n$. Let $p'_i : X \rightarrow C_i$ denote the projection onto the i -th factor. Let $j : D \hookrightarrow X$ be an irreducible curve and let $\phi_n : \tilde{D} \rightarrow D$ denote its normalization. Denote the composite $\phi_n \circ j$ by $\tilde{j} : \tilde{D} \rightarrow X$. Let g denote the genus of \tilde{D} . Let m be an integer such that $m > n, 2g + 2$.

Define a morphism $\psi : \tilde{D}^m \rightarrow X$ by projecting onto the first n coordinates as follows

$$\psi : \tilde{D}^m \rightarrow X \quad \psi := (p'_1 \circ \tilde{j}) \times (p'_2 \circ \tilde{j}) \times \dots \times (p'_n \circ \tilde{j})$$

The morphisms \tilde{j} factor as

$$(2.3) \quad \begin{array}{ccc} & \tilde{D}^m & \\ \Delta \nearrow & \downarrow \psi & \\ \tilde{D} & \xrightarrow{\tilde{j}} & X \end{array}$$

The 1-cycle $\psi_*(\Delta_e)$ on X is smash nilpotent. We want to understand what this 1-cycle looks like. Write the set S as a disjoint union

$$S = S_0 \sqcup S' \quad S_0 = \{1, 2, \dots, n\} \quad S' = \{n+1, \dots, m\}$$

By \underline{x} we will denote closed points of X . For $T \subset S_0$ and \underline{c} , define

$$\zeta_T^{\underline{c}} : X \rightarrow X$$

as follows

$$p_i \circ \zeta_T^{\underline{c}}(\underline{x}) = \begin{cases} x_i & \text{for } i \in T \\ c_i & \text{for } i \notin T \end{cases}$$

For example, if $T = S_0$, then ζ_T^c is the identity. In words, ζ_T^c is the composite of a projection onto the T coordinates and then an inclusion into X using base points given by \underline{c} .

Remark 5. It is clear that if $\underline{v}, \underline{w} \in X$ are two closed points, then for any cycle α , the cycles $\zeta_{T*}^v(\alpha)$ and $\zeta_{T*}^w(\alpha)$ are algebraically equivalent.

Observe that $\psi_*(\Delta_T) = 0$ if $T \cap S_0 = \emptyset$.

Define \underline{v} by $v_i = p'_i \circ \tilde{j}(d_0)$. It follows that

$$\begin{aligned} \psi_*(\Delta_e) &= \psi_*\left(\sum_{T \neq \emptyset, T \subset S} q_{\#T} \Delta_T\right) = \sum_{T \neq \emptyset, T \subset S} q_{\#T} \psi_*(\Delta_T) \\ &= \sum_{T \subset S, T \cap S_0 \neq \emptyset} q_{\#T} \zeta_{(T \cap S_0)*}^v(\tilde{j}_*(\tilde{D})) \end{aligned}$$

One has $T \cap S_0 = S_0$ if and only if $S_0 \subset T$. Let us denote by \mathcal{U} the collection of subsets of S having this property and let us denote by \mathcal{S} the collection of subsets of S with the property $T \cap S_0 \neq \emptyset$. Then we have

$$\begin{aligned} \psi_*(\Delta_e) &= \sum_{T \in \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^v(\tilde{j}_*(\tilde{D})) + \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^v(\tilde{j}_*(\tilde{D})) \\ &= \sum_{T \in \mathcal{U}} q_{\#T} [D] + \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^v([D]) \\ &= [D] \left(\sum_{T \in \mathcal{U}} q_{\#T} \right) + \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^v([D]) \\ (2.4) \quad &= [D] \left(\sum_{i=0}^{m-n} \binom{m-n}{i} q_{i+n} \right) + \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^v([D]) \end{aligned}$$

In equation (2.4) we have put, for $T \in \mathcal{U}$, $\#(T \setminus S_0) = i$, and so $\#T = i + n$. For each i , there are exactly $\binom{m-n}{i}$ choices of T 's in \mathcal{U} with $\#T = i$.

In section 4, lemma 16 it is shown for $n \geq 3$, we can find integers $m > n, 2g + 2$ and rational numbers $q_i, i = 1, 2, \dots, m$ satisfying (S1) and (S2) and

$$(S3) \quad \sum_{i=0}^{m-n} \binom{m-n}{i} q_{i+n} \neq 0$$

Let $\kappa := \sum_{i=0}^{m-n} \binom{m-n}{i} q_{i+n}$, then $\kappa \neq 0$ and from the above we conclude that the following cycle on X is smash nilpotent

$$(2.5) \quad [D] + \frac{1}{\kappa} \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^v([D])$$

Observe that the second term consists of cycles coming from a smaller product of curves. We now prove the main theorem of this section

Theorem 6. *Numerical and smash equivalence coincide for 1-cycles on a product of curves.*

Proof. Let α be a one dimensional cycle on X such that

$$\alpha = \sum_{i=1}^t a_i D_i$$

where the D_i are its distinct irreducible components. Let \tilde{D}_i denote the normalization of D_i and define as above

$$\tilde{j}_i : \tilde{D}_i \rightarrow X$$

Choose a base point $d_i \in \tilde{D}_i$. Using the d_i define points $\underline{v}^i \in X$ by $v_j^i = p_j \circ \tilde{j}_i(d_i)$. Now choose a large integer $m > n, 2g(\tilde{D}_i) + 2$, and corresponding to this a collection of rational numbers such that (S1), (S2) and (S3) are satisfied. Carrying out the preceding discussion for each i , we get as in equation (2.5) that the following cycle is smash nilpotent

$$(2.6) \quad [D_i] + \frac{1}{\kappa} \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^{\underline{v}^i}([D_i])$$

Multiplying with a_i and summing over i , we get

$$\alpha + \frac{1}{\kappa} \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \sum_{i=1}^t a_i \zeta_{(T \cap S_0)*}^{\underline{v}^i}([D_i])$$

is smash nilpotent. Finally, going modulo algebraic equivalence and using remark 5 we get the following cycle is smash nilpotent

$$\begin{aligned} \alpha + \frac{1}{\kappa} \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \sum_{i=1}^t a_i \zeta_{(T \cap S_0)*}^{\underline{v}^i}([D_i]) &= \alpha + \frac{1}{\kappa} \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \sum_{i=1}^t a_i \zeta_{(T \cap S_0)*}^{\underline{v}^1}([D_i]) \\ &= \alpha + \frac{1}{\kappa} \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^{\underline{v}^1} \left(\sum_{i=1}^t a_i D_i \right) \\ &= \alpha + \frac{1}{\kappa} \sum_{T \in \mathcal{S} \setminus \mathcal{U}} q_{\#T} \zeta_{(T \cap S_0)*}^{\underline{v}^1}(\alpha) \end{aligned}$$

Since α is numerically trivial it follows that $\zeta_{(T \cap S_0)*}^{\underline{v}^1}(\alpha)$ is numerically trivial. By induction on n , since $\zeta_{(T \cap S_0)*}^{\underline{v}^1}(\alpha)$ is the pushforward of a cycle from a smaller product of curves, we may assume that it is smash nilpotent. Thus, α being the sum of smash nilpotent cycles, is smash nilpotent. The base case for the induction is the product of two curves, which is a surface. On a surface numerical and algebraic equivalence coincide, and so numerical and smash equivalence coincide, see [Voe95, Cor. 3.3]. \square

Proposition 7. *Let C be a smooth projective curve. Denote by $\alpha_j \in CH_j^{g-1}(J(C))/\sim_{alg}$ its Beauville components. If $\alpha_i = 0$ for some $i \geq 0$, then $\alpha_j = 0$ for every $j > i$.*

Proof. It suffices to show that if $\alpha_i = 0$ then $\alpha_{i+1} = 0$. Recall that the modified diagonal cycle on C^m was defined as

$$\Delta_e = \sum_{T \neq \emptyset, T \subset S} q_{\#T} \Delta_T$$

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where $S = \{1, 2, \dots, m\}$. Since $\alpha_i = 0$, arguing as in lemma 4, it can be checked that Δ_e is algebraically trivial if

$$(2.7) \quad \sum_{k=1}^m q_k \binom{m}{k} k^j = 0 \quad \text{for } j \in \{1, 2, \dots, g-2\} \setminus i$$

Let $\psi : C^m \rightarrow C^n$ denote the projection onto the first n factors, then

$$\psi_*(\Delta_e) = \sum_{l=1}^n \left(\sum_{j=0}^{m-n} \binom{m-n}{j} q_{l+j} \right) \cdot \left(\sum_{\#W=l} \Delta_W \right)$$

Pushing this cycle forward to the Jacobian, we get the cycle

$$(2.8) \quad \sum_{l=1}^n \left(\sum_{j=0}^{m-n} \binom{m-n}{j} q_{l+j} \right) \binom{n}{l} [l]_*([C]) = \sum_{l=1}^n \binom{n}{l} \left(\sum_{j=0}^{m-n} \binom{m-n}{j} q_{l+j} \right) \sum_{s=0}^{g-2} l^{2+s} \alpha_s$$

If the above cycle is algebraically equivalent to 0, then for every k , one of the following two holds

- (a) $\alpha_s \sim_{alg} 0$
- (b) $\sum_{l=1}^n \binom{n}{l} \left(\sum_{j=0}^{m-n} \binom{m-n}{j} q_{l+j} \right) l^{2+s} = 0$

Putting $s = i + 1$ in the above, we may ask if we can find m and rational numbers q_k satisfying the following conditions

- (1) $\sum_{k=1}^m q_k \binom{m}{k} k^j = 0 \quad \text{for } j \in \{1, 2, \dots, g-2\} \setminus i$
- (2) $\sum_{l=1}^n \binom{n}{l} \left(\sum_{j=0}^{m-n} \binom{m-n}{j} q_{l+j} \right) l^{i+3} \neq 0$

This is possible is proved in lemma 17. □

Question. One may ask if the above proposition holds for higher dimensional subvarieties an abelian variety, that is, if $X \hookrightarrow A$ is a subvariety and if $[X]_i$ denote the Beauville components of $[X]$, then does $[X]_i \sim_{alg} 0$ for some $i > 0$ imply $[X]_j \sim_{alg} 0$ for all $j > i$.

3. CYCLES ON ABELIAN VARIETIES

Lemma 8. *Let C be a smooth projective curve of genus g and let $J(C)$ denote its Jacobian. Numerical and smash equivalence coincide for 1-cycles on $J(C)$.*

Proof. Using the same notation as in the previous section, we have

$$C^m \xrightarrow{f} S^m C \xrightarrow{p} J(C)$$

For $m > 2g - 2$, p is a projective bundle associated to a locally free sheaf and so we have for any cycle $\alpha \in CH_{\mathbb{Q}}^*(J(C))$

$$\alpha = \frac{1}{m!} p_* f_* f^* (p^* \alpha \cap c_1(\mathcal{O}(1))^{m-g})$$

Thus, if α is a numerically trivial 1-cycle, then $f^*(p^*\alpha \cap c_1(\mathcal{O}(1))^{m-g})$ is a numerically trivial 1-cycle on C^m and so by the results of the previous section, this cycle is smash nilpotent and so α is smash nilpotent. \square

Theorem 9. *Numerical and smash equivalence coincide for 1-cycles on abelian varieties.*

Proof. Let A be an abelian variety. Then we can find another abelian variety B such that $A \times B$ is isogenous to the Jacobian of a curve. Let $\phi : A \times B \rightarrow J(C)$ and $\psi : J(C) \rightarrow A \times B$ be such that $\psi \circ \phi = [l]_{A \times B}$. Let $\alpha \in CH_s^{g_A-1}(A)$ be a numerically trivial 1-cycle. Consider the inclusion

$$i : A \rightarrow A \times B$$

given by

$$x \mapsto (x, e_B)$$

Since $p_1 \circ i = 1_A$, we get $p_{1*} \circ i_* = 1_{CH(A)}$ and so i_* is an inclusion. Now consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & A \times B \\ [n] \downarrow & & \downarrow [n] \times [n] \\ A & \xrightarrow{i} & A \times B \end{array}$$

Since

$$([n] \times [n])_* i_* \alpha = i_* [n]_* \alpha = n^{2+s} i_* \alpha$$

we get that $i_* \alpha \in CH_s^{g_B+g_A-1}(A \times B)$ and is numerically trivial. Now $\psi^*(i_* \alpha)$ is numerically trivial and thus smash nilpotent, and so $\phi^* \psi^*(i_* \alpha)$ is smash nilpotent. This is $l^{2(g_B+g_A-1)-s} i_* \alpha$ and so we get $i_* \alpha$ is smash nilpotent. Pushing this forward to A , we get that α is smash nilpotent. \square

Proposition 10. *Let A be an abelian variety of dimension g . Let $\theta \in CH_{\mathbb{Q}}^1(A)$ be an ample symmetric class, that is, $[-1]^* \theta = \theta$. The multiplication map $\times \theta^{g-2p+s} : CH_s^p(A) \rightarrow CH_s^{g-p+s}(A)$ is an isomorphism.*

Proof. [Bea, Proposition 5.5]. \square

Proposition 11. *Cycles in $CH_{p-1}^p(A)$ are smash nilpotent for $p > 1$.*

Proof. Let $\alpha \in CH_{p-1}^p(A)$ be a numerically trivial cycle. Let \mathcal{F}_A denote the Fourier transform. Let B denote the abelian variety $A^{\times r}$. Observe that

$$\mathcal{F}_B(\alpha^{\times r}) = \mathcal{F}_A(\alpha)^{\times r}$$

Since $\mathcal{F}_A(\alpha)$ is a numerically trivial 1-cycle, it follows from proposition 9 that it is smash nilpotent. In particular, using the above equality, for some $r > 0$ we get that $\mathcal{F}_B(\alpha^{\times r}) = 0$ and since \mathcal{F} is an isomorphism, $\alpha^{\times r} = 0$ \square

Following [Fak96] we denote by $\text{Griff}_s^p(A) := CH_s^p(A) / \sim_{alg}$.

Theorem. [Fak96, Theorem 4.4] *$\text{Griff}_2^i(A)$ ($i = 3, 4$) is infinite dimensional for the generic abelian variety A of dimension 5.*

Corollary. [Fak96, Corollary 4.6] *$\text{Griff}_2^{g-1}(J) \neq 0$ for the generic Jacobian J of dimension $g \geq 11$.*

Corollary 12. *The above cycles are smash nilpotent.*

Proof. Follows from proposition 11 and theorem 9. \square

Let C be a smooth projective curve. Let $A(J(C))$ denote the \mathbb{Q} algebra of cycles modulo algebraic equivalence on $J(C)$. The tautological subring of $A(J(C))$, see [Bea04], is defined as the \mathbb{Q} subspace of $A(J(C))$ generated by the class of the curve (which is well defined since we are working modulo algebraic equivalence) under the operations : intersection product, Pontryagin product, pullback and pushforward under multiplication by integers.

Corollary 13. *Numerical and smash equivalence coincide on the tautological subring of $A(J(C))$.*

Proof. Let g denote the genus of the curve. Let us denote the tautological subring by R . Since R is stable under pullback by multiplication by integers, it has a decomposition as a sum of its Beauville components. The intersection product and Pontryagin product have the property

$$A_s^p \cdot A_{s'}^{p'} \subset A_{s+s'}^{p+p'} \quad A_s^p * A_{s'}^{p'} \subset A_{s+s'}^{p+p'-g}$$

We follow Beauville's notation and denote the class of the curve by $[C] = \sum_{i=0}^{g-1} C_{(i)}$ where $C_{(i)} \in A_i^{g-1}(J(C))$. Since R is generated by the class of the curve, we get

$$R = \bigoplus_p R^p = \bigoplus_p \bigoplus_{s \geq 0}^p R_s^p = \left(\bigoplus_p R_0^p \right) \oplus \left(\bigoplus_p \bigoplus_{s > 0}^p R_s^p \right)$$

The summand $\bigoplus_p \bigoplus_{s > 0}^p R_s^p$ is an ideal in R , with respect to the intersection product and Pontryagin product, which we denote by I .

Let $w^{g-d} := (1/d!)C^{*d} \in A^{g-d}(J(C))$. The $N^i(w)$ are defined as the Newton polynomials in the classes w^i . For more details we refer the reader to [Bea04]. We will use the following results from [Bea04],

- R is generated as a \mathbb{Q} -subalgebra, under the intersection product, by $N^i(w)$ (see the line after [Bea04, Corollary 3.5]).
- In [Bea04, Corollary 3.4] it is shown that $N^i(w) = -\mathcal{F}(C_{(i-1)})$ for $i \geq 1$. In particular, for $i \geq 2$ one gets $N^i(w) \in I$.

Since $N^1(w) = \theta$, from the above it is clear that $\bigoplus_p R_0^p$ is the vector space spanned by $\langle \theta^p \rangle$. Thus, $0 \neq \beta \in R$ is numerically trivial iff $\beta \in I$. Since I is generated by $N^i(w) = -\mathcal{F}(C_{(i-1)})$ for $i \geq 2$ we get that β is smash nilpotent. \square

4. SOLVING THE EQUATIONS

Consider the following set of linear homogeneous polynomials in the q 's

- (1) $l_1 = \sum_{k=1}^m \binom{m}{k} k q_k$
- (2) $l_{2,2i} = \sum_{k=1}^m \binom{m}{k} k^{2i} q_k$, for every i in the set $\{1, \dots, t\}$ (think of t as $\lfloor \frac{g+1}{2} \rfloor$)
- (3) $l_3 = \sum_{k=n}^m \binom{m-n}{k-n} q_k$

It will suffice for us if we can find m 's and q 's such that the following systems have solutions

$$(4.1) \quad l_1 = 0, l_{2,2i} = 0, l_3 \neq 0$$

Consider the following collection of polynomials in one variable.

$$(1) \quad r_i(x) = \sum_{k=1}^m \binom{m}{k} k^i x^k$$

$$(2) \quad x^n(1+x)^{m-n} = \sum_{k=n}^m \binom{m-n}{k-n} x^k$$

If T denotes the operator $(x \frac{d}{dx})$ then

$$r_i(x) = T^i(1+x)^m$$

Hence, it would suffice for us if we can show that the following polynomials are linearly independent for $2t < m$

$$\left\{ x^n(1+x)^{m-n}, \quad r_1(x), \quad r_{2i}(x)_{i=1,\dots,t} \right\}$$

We claim this is true for $3 \leq n \leq m$. If not, then \exists a nontrivial linear dependence

$$(4.2) \quad \beta_1 T(1+x)^m + \sum_{i=1}^t \beta_{2i} T^{2i}(1+x)^m + a x^n(1+x)^{m-n} = 0$$

Define $m[j] := m(m-1) \cdots (m-j)$.

Lemma 14. *We have the following expression for $1 \leq i \leq m$*

$$T^i(1+x)^m = \sum_{j=1}^i c_j^i m[j-1] x^j (1+x)^{m-j}$$

where the coefficients c_j^i satisfy the relation $c_j^{i+1} = j c_j^i + c_{j-1}^i$.

Proof. For $i = 1$, $c_1^1 = 1$ and the statement is clear. We prove the statement by induction on i . In the following calculation, put $c_j^i = 0$ for $i < j < 1$. If D denotes the operator $\frac{d}{dx}$, then

$$\begin{aligned} DT^i(1+x)^m &= \sum_{j=1}^i c_j^i m[j-1] D(x^j(1+x)^{m-j}) \\ &= \sum_{j=1}^i c_j^i m[j-1] \left(j x^{j-1} (1+x)^{m-j} + (m-j) x^j (1+x)^{m-j-1} \right) \\ xDT^i(1+x)^m &= \sum_{j=1}^i c_j^i j m[j-1] x^j (1+x)^{m-j} + \sum_{j=1}^i c_j^i m[j] x^{j+1} (1+x)^{m-j-1} \\ T^{i+1}(1+x)^m &= \sum_{j=1}^{i+1} \left(j c_j^i + c_{j-1}^i \right) m[j-1] x^j (1+x)^{m-j} \end{aligned}$$

□

Remark 15. It is clear that the coefficients c_j^i are independent of m .

From the above lemma it follows that, for $i \leq m$, the highest power of $(1+x)$ which divides $T^i(1+x)^m$ is $m-i$. Looking at the least power of $(1+x)$ which divides all the terms in equation (4.2), we get that $\beta_i = 0$ for $m-2i < m-n$, that is, for $i > \frac{n}{2}$. Thus, we may rewrite the relation as

$$(4.3) \quad \beta_1 T(1+x)^m + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \beta_{2i} T^{2i}(1+x)^m = -ax^n(1+x)^{m-n}$$

This shows that

$$(4.4) \quad x^n \left| \beta_1 T(1+x)^m + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \beta_{2i} T^{2i}(1+x)^m \right.$$

Next we will show that equation (4.4) leads to a contradiction.

Lemma 16. *The following polynomials are linearly independent for $m > n \geq 3$, $2t < m$,*

$$\left\{ x^n(1+x)^{m-n}, \quad r_1(x), \quad r_{2i}(x)_{i=1, \dots, t} \right\}$$

Proof. Equating to 0 the coefficient of x^k , for $k \in \{1, 2, \dots, n-1\}$, in

$$\beta_1 T(1+x)^m + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \beta_{2i} T^{2i}(1+x)^m = \beta_1 r_1(x) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \beta_{2i} r_{2i}(x)$$

we get

$$(4.5) \quad \beta_1 k + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \beta_{2i} k^{2i} = 0 \quad k \in \{1, 2, \dots, n-1\}$$

It is clear that all $\beta_i = 0$ if $n-1 \geq \lfloor \frac{n}{2} \rfloor + 1$, that is, for $n \geq 3$. \square

Lemma 17. *For $1 \leq i < n-3$, $\exists m \gg 0$ such that the polynomial $(1+x)^{m-n} T^{i+3}(1+x)^n$ is not a rational linear combination of $r_j(x)$ for $j \in \{1, \dots, n\} \setminus i$*

Proof. Assume that

$$(1+x)^{m-n} T^{i+3}(1+x)^n = \sum_{j \in \{1, \dots, n\} \setminus i} \beta_j T^j(1+x)^m$$

Using lemma 14 and comparing the highest power of $(1+x)$ dividing all terms in the above equation, we get $\beta_j = 0$ for $j > i+3$. Thus,

$$(1+x)^{m-n} T^{i+3}(1+x)^n = \sum_{j \in \{1, \dots, i+3\} \setminus i} \beta_j T^j(1+x)^m$$

Again using lemma 14 and comparing the highest powers of $(1+x)$ on both sides of the above equality, we get, for $j \in \{i+3, i+2, i+1, i\}$, the following equations

$$(4.6) \quad \begin{aligned} c_{i+3}^{i+3} n[i+2] &= \beta_{i+3} c_{i+3}^{i+3} m[i+2] \\ c_{i+2}^{i+3} n[i+1] &= \beta_{i+2} c_{i+2}^{i+2} m[i+1] + \beta_{i+3} c_{i+2}^{i+3} m[i+1] \\ c_{i+1}^{i+3} n[i] &= \beta_{i+1} c_{i+1}^{i+1} m[i] + \beta_{i+2} c_{i+1}^{i+2} m[i] + \beta_{i+3} c_{i+1}^{i+3} m[i] \\ c_i^{i+3} n[i-1] &= \beta_{i+1} c_i^{i+1} m[i-1] + \beta_{i+2} c_i^{i+2} m[i-1] + \beta_{i+3} c_i^{i+3} m[i-1] \end{aligned}$$

Solving for $\beta_{i+3}, \beta_{i+2}, \beta_{i+1}$ using the first three of the above equations we get

$$\begin{aligned}\beta_{i+3} &= \frac{n[i+2]}{m[i+2]} \quad (\because c_i^i = 1 \quad \forall i \geq 1) \\ \beta_{i+2} &= c_{i+2}^{i+3} \left(\frac{n[i+1]}{m[i+1]} - \frac{n[i+2]}{m[i+2]} \right) \\ \beta_{i+1} &= c_{i+1}^{i+3} \left(\frac{n[i]}{m[i]} - \frac{n[i+2]}{m[i+2]} \right) - c_{i+1}^{i+2} c_{i+2}^{i+3} \left(\frac{n[i+1]}{m[i+1]} - \frac{n[i+2]}{m[i+2]} \right)\end{aligned}$$

Substituting the above values into (4.6), we get

$$\begin{aligned}c_i^{i+3} \frac{n[i-1]}{m[i-1]} &= c_i^{i+1} c_{i+1}^{i+3} \left(\frac{n[i]}{m[i]} - \frac{n[i+2]}{m[i+2]} \right) - c_i^{i+1} c_{i+1}^{i+2} c_{i+2}^{i+3} \left(\frac{n[i+1]}{m[i+1]} - \frac{n[i+2]}{m[i+2]} \right) \\ &\quad + c_i^{i+2} c_{i+2}^{i+3} \left(\frac{n[i+1]}{m[i+1]} - \frac{n[i+2]}{m[i+2]} \right) + c_i^{i+3} \frac{n[i+2]}{m[i+2]}\end{aligned}$$

For $r < s$ denote by $m[s, r] := \frac{m[s]}{m[r]}$. With this notation the above equation is

$$\begin{aligned}c_i^{i+3} &= c_i^{i+1} c_{i+1}^{i+3} \left(\frac{n[i, i-1]}{m[i, i-1]} - \frac{n[i+2, i-1]}{m[i+2, i-1]} \right) \\ &\quad - c_i^{i+1} c_{i+1}^{i+2} c_{i+2}^{i+3} \left(\frac{n[i+1, i-1]}{m[i+1, i-1]} - \frac{n[i+2, i-1]}{m[i+2, i-1]} \right) \\ &\quad + c_i^{i+2} c_{i+2}^{i+3} \left(\frac{n[i+1, i-1]}{m[i+1, i-1]} - \frac{n[i+2, i-1]}{m[i+2, i-1]} \right) + c_i^{i+3} \frac{n[i+2, i-1]}{m[i+2, i-1]}\end{aligned}$$

If we keep i, n fixed and let $m \rightarrow \infty$, then the RHS is 0, while the LHS is a fixed positive integer, which is a contradiction. \square

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